Relative Calabi–Yau structure on microlocalization (joint work with Chris Kuo)

AMS 2024 Fall Sectional Meeting

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Calabi–Yau structures

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- Similar duality results appear in smooth topology. An orientation on a closed (smooth) manifold is a volume form, which induces the isomorphism $H^*(M) = H_{n-*}(M)$. For local systems on closed oriented manifolds,

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- One can define the abstract property for general dg-categories.

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Definition

Let A be a proper dg-category (which means $A(X,Y) = Hom_A(X,Y)$ is finite dimensional). Define the dualizing bimodule by $A^\vee(X,Y) = Hom(Y,X)^\vee$. A proper Calabi–Yau structure on A is a class in $HC_*(A)^\vee$ that induces the bimodule isomorphism $A \simeq A^\vee[-n]$.

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Definition (Kontsevich–Soibelman)

Let A be a smooth dg-category (which means the diagonal bimodule A has a finite resolution by Yoneda bimodules). Define the inverse dualizing bimodule by $A^!(X, Y) = Hom_{A \otimes A^{op}}(A, A \otimes A)$. A smooth Calabi–Yau structure on A is a class in $HC_*^-(A)$ that induces the bimodule isomorphism $A^! \simeq A[-n]$.

Relative Calabi–Yau structures

- For a (smooth) Fano variety X with an anticanonical divisor D, a log Calabi–Yau structure is a holomorphic volume form $K_X = \mathcal{O}_X(-D)$. The long exact sequence and Serre duality implies

$$Hom(F, G \otimes \mathcal{O}_X(-D)) \to Hom(F, G) \to Hom(i^*F, i^*G) \xrightarrow{+1},$$
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- Similar duality long exact sequences appear in smooth topology. For a compact manifold M with boundary ∂M, an orientation is a volume form on $H^n(M, \partial M)$. Poincaré–Lefschetz duality implies

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- We can also generalize the structure to dg-functors between dg-categories.

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Definition (Brav–Dyckerhoff)

Let A and B be a smooth dg-categories and $F : B \to A$ be a dg functor. A relative smooth Calabi–Yau structure on F is a class in the relative negative cyclic homology $HC_*^-(B, A) = \mathrm{Cone}(HC_*^-(B) \to HC_*^-(A))$ that induces an exact triangle of bimodule isomorphisms

$$\begin{array}{ccccccc}
A^![1-n] & \longrightarrow & F_! B^![1-n] & \longrightarrow & \mathrm{Cone}^![-n] & \xrightarrow{+1} & \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
\mathrm{Cone}[-1] & \longrightarrow & F_! B & \longrightarrow & A & \xrightarrow{+1} &
\end{array}$$

Fukaya categories

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- Fukaya categories are invariants of symplectic manifolds, whose objects are certain Lagrangian submanifolds (boundary conditions) and whose morphisms are infinite dimensional Morse theory of Lagrangian intersections.
- Comparing to coherent sheaves of (commutative) algebraic varieties, the Fukaya categories include representation categories of many non-commutative algebras.

Microlocal sheaves and Fukaya categories

- Nadler–Zaslow noticed that the Fukaya categories on cotangent bundles T^*M should be equivalent to the categories of constructible sheaves on the base manifold M. (Constructible sheaves are sheaves that are locally constant with respect to some stratifications.)

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- Moreover, the singular support of constructible sheaves, which are singular conic Lagrangians in T^*M (they are subsets of the union of conormals of the stratification) correspond to the Lagrangian submanifolds after taking limits.

Microlocal sheaves and Fukaya categories

- Following the proposal of Kontsevich, Nadler–Shende and Ganatra–Pardon–Shende generalized the result to a larger class of open symplectic manifolds. They showed that the Fukaya categories are (co)sheaves on a subset called the Lagrangian skeleton of the symplectic manifold, called microlocal sheaves.

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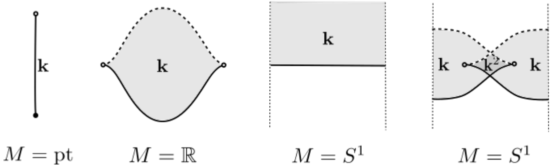
Theorem (Kuo–L.)

For a Legendrian $\Lambda \subset S^*M$, the category of sheaves with Legendrian singular supports $Sh_\Lambda(M)$ and the category of microlocal sheaves on the Legendrian boundary $\mu sh_\Lambda(\Lambda)$ form a relative Calabi–Yau pair.

Microlocal sheaves and Fukaya categories

- Consider the projection of the Legendrian $\pi(\Lambda) \subset M$. Sheaves in $Sh_\Lambda(M)$ are constructible with respect to the stratification by $\pi(\Lambda) \subset M$, with maps from low dimensional strata to high dimensional strata.



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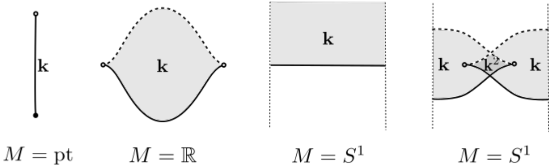

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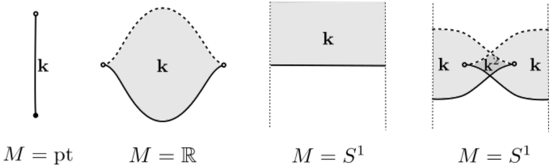
- For the maps from low dimensional strata to high dimensional ones along the codirection of Λ, the mapping cone determines the stalk of the object in $\mu sh_\Lambda(\Lambda)$ (called microstalks).



Discussion of our main result

- We can identify the inverse dualizing bimodule with an explicit geometric construction, given by wrapping the Legendrian $\Lambda \subset S^*M$ around positive flows once.

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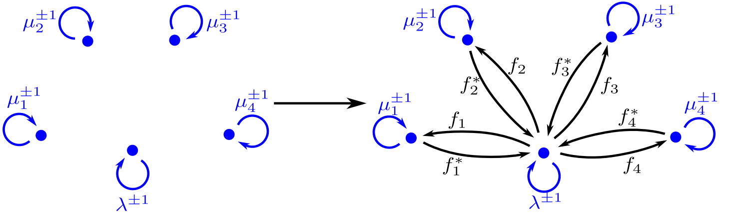
- There are works in progress about weak relative Calabi–Yau structures on Legendrian contact homologies (whose module categories are equivalent to wrapped Fukaya categories), by Z. Chen, G. Dimitroglou Rizell–N. Legout, J. Sabloff–J. Ma.

Examples of Calabi–Yau pairs by our result

- The derived relative multiplicative preprojective algebras, which model perverse sheaves on a disk with stratifications by n points.



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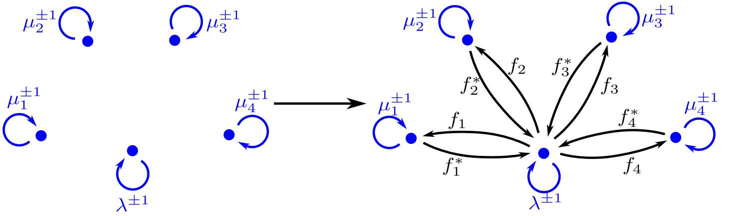

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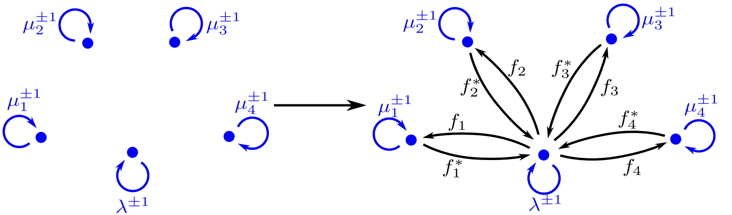
- Relative Ginzburg algebras or relative Calabi–Yau completions (however, one needs to go beyond cotangent bundles).



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- (Potentially) coproducts on the free loop space is determined by the Calabi–Yau structure, and distinguish simple homotopy types of manifolds.

Thank you!